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ATTRACTIVITY FOR PANTOGRAPH EQUATIONS WITH ψ-TYPE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

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ABSTRACT

The key purpose of the present work is to examine a fractional-order pantograph equation to ψ-type fractional derivative in Riemann-Liouville sense. The existence of globally attractive solutions for fractional-order pantograph equations is discussed.

Keywords: ψ-fractional derivative; Pantograph equations; Attractivity; Measure of noncompactness.

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1. INTRODUCTION

The pantograph equation is a special type of functional differential equations with pro-portional delay. It arises in rather different fields of pure and applied mathematics, such as electrodynamics, control systems, number theory, probability, and quantum mechanics. Many researchers have studied the pantograph-type delay differential equation using analytical and numerical techniques [2, 10, 11, 12].

Fractional calculus is applied in different directions of physics, mathematical biology, fluid mechanics, electrochemistry, signal processing, viscoelasticity, and finance and in many more. In the branch of fractional calculus, fractional derivatives and fractional integrals are important aspects. Recently, many researchers and scientists have analyzed issues in this special branch [6, 7, 8, 9]; The aim of this paper is to study the existence and global attractivity of solutions for a pantograph equation of fractional order. The main technique used in our considerations is the measures of noncompactness and a fixed point theorem of generalized Ascoli-Arzela theorem. Our investigations will be situated in the Banach space of real functions which are defined, continuous, and bounded. The work on the attractivity of solutions for fractional differential equations in Banach space was initiated in [5]. Motivated by this work, here we study the question of attractivity of solutions for a class of pantograph equations in the sense of ψ-type Riemann-Liouville (R-L) fractional derivative given by

$$\begin{cases} ({}^{L}D^{\alpha;\psi}y)(t) = f(t, y(t), y(\lambda t)), \ t \in (0, \infty), \\ (I^{1-\alpha;\psi}y)(0) = y_0, \end{cases} \tag{1}$$

where $0 < \lambda < 1$, ${}^{L}D^{\alpha;\psi}$ is ψ-type R-L fractional derivative of order $\alpha \in (0, 1)$, $\mathrm{I}^{1-\alpha;\psi}$ is ψ-type R-L fractional integral of order $1 - \alpha$, $f : [0, \infty) \times Y \times Y \to Y$ is a continuous function satisfying some hypotheses and y_0 is an element of the Banach space Y.

The development of this article is as follows. In Section 2, the ψ-type fractional derivative is discussed. In Section 3, we estabilish sufficient conditions for the global attractivity for solutions of problem (1).

2. PREREQUISITES

In the present part, we give some definitions and properties of the fractional derivative as suggested by [1].

Let $\alpha \in (0, 1)$ and $x \in L^1([0, \infty), Y)$. The ψ-type R-L integral is defined by

$$(I^{\alpha;\psi} x(t) = g_\alpha(t) *_\psi x(t) = \textstyle\int_0^t g_\alpha(t-s)x(s)\psi'(s)ds, \ t > 0,$$

where * denotes the convolution,

$$g_\alpha(t) = \frac{(\psi(t))^{\alpha-1}}{\Gamma(\alpha)}.$$

For $x \in C([0, \infty), Y)$, the ψ-type R-L fractional derivative is defined by

$$({}^{L}D^{\alpha;\psi}x)(t) = \frac{d}{dt}(g_{1-\alpha}(t) *_\psi x(t)).$$

Lemma 2.1: [10] Assume that the operator $f : [0, \infty) \times Y \times Y \to Y$ is continuous. The problem (1) is equivalent to the integral equation

$$y(t) = (\psi(t))^{\alpha-1} y_0 + \int_0^t \psi'(t)(\psi(t) - \psi(s))^{\alpha-1} f(s, y(s), y(\lambda s))ds,\ t > 0, \tag{2}$$

provided the right side is point-wise defined on $(0, \infty)$.

We introduce terminology, let

$$C_0([t_0,\infty),Y) = \{y \in C([t_0,\infty),Y) : \lim_{t\to\infty} |y(t)| = 0\}.$$

It is obviously that $C_0([t_0, \infty), Y)$ is a Banach space.

We need also following gneralized Ascoli-Arzela theorem [4].

Lemma 2.2: The set $H \subset C_0([t_0, \infty), Y)$ is relatively compact if and only if the following conditions hold:

1. for any $T > 0$, the functions in H are equicontinuous on $[0, T]$;
2. for any $t \in [0, \infty)$, $H(t) = \{y(t) : y \in H\}$ is relatively compact in Y;
3. $\lim_{t\to\infty} |y(t)| = 0$ uniformly for $y \in H$.

3. MAIN RESULTS

We introduce the following hypotheses:

(H1) $|f(t, y, y)| \le L(\psi(t))^{-\beta} |y|^{\delta}$ for $t \in (0, \infty)$ and $y \in Y$, $L \ge 0$, $\alpha < \beta < 1$ and $\delta \in \mathbb{R}$.

(H2) There exists a constant $\kappa > 0$ such that for any bounded set $E \subset Y$,

$$\alpha(f(t, E, E)) \le \kappa\sigma(E),$$

where σ is the Hausdorff measure of noncompactness.

For any $y \in C([0, \infty), Y)$ and a given $n \in \mathbb{N}^+$, define an operator U as follows:

$$(Uy)(t) = \left((\psi(t)) + \frac{1}{n}\right)^{\alpha+1} y_0 + \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, y(s), y(\lambda s))ds,$$
$$t \in [0, \infty). \tag{3}$$

Since $0 < \alpha < \beta < 1$, we can choose a $\gamma > 0$ sufficiently small such that

$$\alpha + \gamma - 1 < 0,\ 1 - \beta - \gamma\beta > 0 \text{ and } \alpha + \gamma - \gamma\delta < 0.$$

Let $T > 0$ sufficiently large such that

$$|y_0| = \left(\psi(t) + \frac{1}{n}\right)^{\alpha+\gamma-1} + \frac{L\Gamma(\alpha)\Gamma(1-\beta-\gamma\beta)}{\Gamma(1+\alpha-\beta-\gamma\delta)}(\psi(t))^{\alpha+\gamma-\beta-\gamma\delta} \le 1, \text{ for } t \ge T. \tag{4}$$

Define a set S as follows

$$S = \{y(t) | y \in C([0, \infty), Y),\ |(\psi(t))^{\gamma} y(t)| \le 1, \text{ for } t \ge T\}.$$

It is easy to see that $S \neq \varnothing$ and S is a closed, convex and bounded subset of C_0 ([0, ∞), Y).

Lemma 3.1: Assume that (H1) holds. Then $\{Uy : y \in S\}$ is equicontinuous and $\lim_{t\to\infty} |(Uy)(t)| = 0$ uniformly for $y \in S$.

Proof: Since $\alpha - \beta - \gamma\delta < 0$, there exists a $T_1 > 0$ sufficiently large such that

$$\left((\psi(t)) + \frac{1}{n}\right)^{\alpha-1} |y_0| < \frac{\in}{4}; \ \frac{L\Gamma(\alpha)\Gamma(1-\beta-\gamma\delta)}{\Gamma(1+\alpha-\beta-\gamma\delta)}(\psi(t))^{\alpha-\beta-\gamma\delta} < \frac{\in}{4}, \text{ for } t \geq T_1.$$

For any $y \in S$, and $t_1, t_2 \geq T_1$, we get

$$\begin{aligned}
&|(Uy)(t_2) - (Uy)(t_2)| \\
&\leq \left((\psi(t_2)) + \frac{1}{n}\right)^{\alpha-1} |y_0| + \left((\psi(t_1)) + \frac{1}{n}\right)^{\alpha-1} |y_0| \\
&+ \int_0^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} |f(s, y(s), y(\lambda s))| ds \\
&+ \int_0^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1} |f(s, y(s), y(\lambda s))| ds \\
&\leq \left((\psi(t_2)) + \frac{1}{n}\right)^{\alpha-1} |y_0| + \left((\psi(t_1)) + \frac{1}{n}\right)^{\alpha-1} |y_0| \\
&+ L + \int_0^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} (\psi(s))^{-\beta-\gamma\delta} ds \\
&+ L + \int_0^{t_2} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1} (\psi(s))^{-\beta-\gamma\delta} ds \\
&\leq \left((\psi(t_2)) + \frac{1}{n}\right)^{\alpha-1} |y_0| + \left((\psi(t_1)) + \frac{1}{n}\right)^{\alpha-1} |y_0| \\
&+ \frac{L\Gamma(\alpha)\Gamma(1-\beta-\gamma\delta)}{\Gamma(1+\alpha-\beta-\gamma\delta)}(\psi(t_2))^{\alpha-\beta-\gamma\delta} + \frac{L\Gamma(\alpha)\Gamma(1-\beta-\gamma\delta)}{\Gamma(1+\alpha-\beta-\gamma\delta)}(\psi(t_1))^{\alpha-\beta-\gamma\delta}.
\end{aligned}$$

Furthermore, for $0 \leq t_1 < t_2 \leq T_1$, we obtain

$$\begin{aligned}
&|(Uy)(t_2) - (Uy)(t_2)| \\
&\leq \left| \left((\psi(t_2)) + \frac{1}{n}\right)^{\alpha-1} - \left((\psi(t_1)) + \frac{1}{n}\right)^{\alpha-1} \right| |y_0|
\end{aligned}$$

$$+\left|\int_0^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} f(s,y(s),y(\lambda s))ds\right.$$

$$\left.-\int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1} f(s,y(s),y(\lambda s))ds\right|$$

$$\leq \left|\left((\psi(t_2))+\frac{1}{n}\right)^{\alpha-1} - \left((\psi(t_1))+\frac{1}{n}\right)^{\alpha-1}\right| |y_0|$$

$$+M\int_0^{t_2} [\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1} - \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}]ds$$

$$+M\int_0^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} ds$$

$$\leq \left|\left((\psi(t_2))+\frac{1}{n}\right)^{\alpha-1} - \left((\psi(t_1))+\frac{1}{n}\right)^{\alpha-1}\right| |y_0|$$

$$+\frac{1}{\alpha}M[(\psi(t_1))^{\alpha} - (\psi(t_2))^{\alpha} + ((\psi(t_2))-(\psi(t_1)))^{\alpha}] + \frac{1}{\alpha}M((\psi(t_2))-(\psi(t_1)))^{\alpha}$$

$$\to 0, \text{ as } t_2 \to t_1,$$

where $M = \sup_{t\in[0,t_2], y\in S} |f(t, y(t), y(\lambda t))|$.

For any $t_1 < T_1 < t_2$, similarly, we obtain

$$|(Uy)(t_2) - (Uy)(t_1)| \leq |(Uy)(t_2) - (Uy)(T_1)| + |(Uy)(T_1) - (Uy)(t_1)|$$
$$\to 0, \text{ as } t_2 \to t_1.$$

Therefore, combining the above argument, it is clear that the family of functions $\{Uy : y \in S\}$ is equicontinuous.

It remains to verify that $\lim_{t\to\infty} |(Uy)(t)| = 0$ uniformly for $y \in S$. Indeed, we have

$$|(Uy)(t)| = \left|\left((\psi(t))+\frac{1}{n}\right)^{\alpha-1} y_0 + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,y(s),y(\lambda s))ds\right|$$

$$\leq \left((\psi(t))+\frac{1}{n}\right)^{\alpha-1} |y_0| + L\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(\psi(s))^{-\beta-\lambda\delta} ds$$

$$\leq \left((\psi(t))+\frac{1}{n}\right)^{\alpha-1} |y_0| + \frac{L\Gamma(\alpha)\Gamma(1-\beta-\gamma\delta)}{\Gamma(1+\alpha-\beta-\gamma\delta)}(\psi(t))^{\alpha-\beta-\gamma\delta}$$

$$\to 0, \text{ as } t \geq T.$$

This shows that $\lim_{t\to\infty} |(Uy)(t)| = 0$ uniformly for $y \in S$. The proof is completed.

Lemma 3.2: Assume that (H1) holds. Then U maps S into S and U is continuous in S.

Proof:

Claim 1: U maps S into S.

For $y \in S$, by Lemma 3.1, we know $Uy \in C([0, \infty), Y)$. On the other hand, for $t \geq T$, by the inequality (4), we have

$$|(\psi(t)^{\gamma})(Uy)(t)|$$

$$\leq (\psi(t))^{\gamma}\left(\left((\psi(t))+\frac{1}{n}\right)^{\alpha-1}|y_0|+\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|f(s,y(s),y(\lambda x))|ds\right)$$

$$\leq\left((\psi(t))+\frac{1}{n}\right)^{\alpha+\gamma-1}|y_0|+L(\psi(t))^{\gamma}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(\psi(s))^{-\beta-\gamma\delta}ds$$

$$\leq\left((\psi(t))+\frac{1}{n}\right)^{\alpha+\gamma-1}|y_0|+\frac{L\Gamma(\alpha)\Gamma(1-\beta-\gamma\delta)}{\Gamma(1+\alpha-\beta-\gamma\delta)}(\psi(t))^{\alpha-\gamma-\beta-\gamma\delta}$$

$$\leq 1,\ t\geq T.$$

which implied that $US \subset S$.

Claim 2: U is continuous in S.

For any $y_m, y \in S$, $m = 1, 2, \ldots$ with $\lim_{t\to\infty} y_m = y$, we will show that $Uy_m \to Uy$, as $m \to \infty$. For $\forall \in > 0$, there exists a $T_2 > 0$ sufficiently large such that

$$\frac{L\Gamma(\alpha)\Gamma(1-\beta-\gamma\delta)}{\Gamma(1+\alpha-\beta-\gamma\delta)}(\psi(T_2))^{\alpha-\gamma-\beta-\gamma\delta}<\frac{\in}{2}.$$

Then, for $t > T_2$, we get

$$|(Uy_m)(t) - (Uy)(t)|$$

$$\leq\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left(|f(s,y_m(s),y_m(\lambda s))|+|f(s,y(s),y(\lambda s))|\right)ds$$

$$\leq 2L\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(\psi(s))^{-\beta-\gamma\delta}ds$$

$$\leq\frac{2L\Gamma(\alpha)\Gamma(1-\beta-\gamma\delta)}{\Gamma(1+\alpha-\beta-\gamma\delta)}(\psi(T_2))^{\alpha-\gamma-\beta-\gamma\delta}<\in.$$

For $0 < t \le T_2$, by lebesgue dominated convergence theorem, we have

$$|(Uy_m)(t) - (Uy)(t)|$$

$$\le \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left(\left|f(s,y_m(s),y_m(\lambda s))\right| + \left|f(s,y(s),y(\lambda s))\right|\right)ds$$

$$\to 0 \text{ as } m \to \infty.$$

Therefore, it is obvious that $\|(Uy_m)-(Uy)\| \to 0$, as $m \to \infty$, which imples that the operator U is continuous. This proof is completed.

Theorem 3.3: Assume that (H1) and (H2) hold. Then the problem (1) admits at least one attractive solution.

Proof: By Lemma 3.2, we know that $U : S \to S$ is bounded and continuous. Next, it will be shown that $U \subset C_0$ ([0, ∞), Y) is relatively compact. By Lemma 3.1, we know that $\{Uy : y \in S\}$ is equicontinuous and $\lim_{t\to\infty} |Uy(t)| = 0$ uniformly for $y \in S$. It remains to verify that for any $t \in [0, \infty)$, $\{(Uy)(t) : y \in S\}$ is relatively compact in Y by using (H2). We omit the proof of this step as it is similar to that of Theorem 3.1 in [5]. Therefore, by Schauder's fixed point theorem, the operator U has a fixed point $y_n \in S$, with $y_n(t) \to 0$ as $t \to \infty$. By using the similar method as in the proof of Lemma 3.1, we know that $\{y_n(t)\}$ is uniformly bounded and equicontinuous on $[0, \infty)$, and for any $t \in [0, \infty)$, $\{y_n(t)\}$ is relatively compact. Therefore, by Arzela-Ascoli's theorem, $\{y_n(t)\}$ has a uniformly converegent subsequence $\{y_{n_k}\}$. Moreover, $\{y_{n_k}\}$ satisfies

$$y_{n_k}(t) = \left((\psi(t)) + \frac{1}{n_k}\right)^{\alpha-1} y_0 + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, y_{n_k}(s), y_{n_k}(\lambda s))ds,$$

$$t \in [0, \infty). \tag{5}$$

When $t \ne 0$, let $y^*(t) = \lim_{k\to\infty} y_{n_k}(t)$. By Lebesgue dominated convergence theorem and (3), we get

$$y^*(t) = (\psi(t))^{\alpha-1} y_0 + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, y^*(s), y_{n_k}(\lambda s))ds,$$

$$t \in [0, \infty).$$

which implies that $y(t)$ is an attractive solution of problem (1). The proof is completed.

In the case where $Y = \mathbb{R}^n$, we have the following corollary which improve the result in [3].

Corollary 3.4: Assume that (H1) holds. Then the problem (1) admits at least one attractive solution.

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AUTHORS CONTRIBUTIONS

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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